Journal of Algebra 321 (2009) 667–681

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Polynomial identities for graded tensor products of algebras

José A. Freitas[1], Plamen Koshlukov[*,2]

IMECC, UNICAMP, P.O. Box 6065, 13083-970 Campinas, SP, Brazil

ARTICLE INFO

Article history:
Received 30 April 2008
Available online 29 October 2008
Communicated by J.T. Stafford

Keywords:
Graded identities
Graded tensor product
Graded T-ideal
T-prime algebras
Commutation factors

ABSTRACT

Let K be a field, $\operatorname{char} K = 0$. We study the polynomial identities satisfied by $\mathbb{Z}_2$-graded tensor products of T-prime algebras. Regev and Seeman proved that in a series of cases such tensor products are PI equivalent to T-prime algebras; they conjectured that this is always the case. We deal here with the remaining cases and thus confirm Regev and Seeman's conjecture. For some "small" algebras we can remove the restriction on the characteristic of the base field, and we show that the behaviour of the corresponding graded tensor products is quite similar to that for the usual (ungraded) tensor products. Finally we consider β-graded tensor products (also called commutation factors) and their identities. We show that Regev's $A \otimes B$ theorem holds for β-graded tensor products whenever the gradings are by finite abelian groups. Furthermore we study the PI equivalence of β-graded tensor products of T-prime algebras.

Introduction

All algebras and vector spaces considered in this paper are over a fixed infinite field K, $\operatorname{char} K \neq 2$. Let G be an additive abelian group. The algebra A is G-graded if $A = \bigoplus_{g\in G} A_g$ is a direct sum of the vector subspaces A_g such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. We give here a couple of important examples of graded algebras. Let V be a vector space, $\dim V = \infty$ and assume $e_1, e_2, \dots$ is a basis of V. The Grassmann algebra $E = E(V)$ is the vector space with basis $\varepsilon = \{e_{i_1} e_{i_2} \dots e_{i_k} \mid k \geqslant 0\}$ where $i_1 < i_2 < \dots < i_k$; if $k = 0$ we denote the corresponding element by 1. The multiplication in V is induced by $e_i e_j = -e_j e_i$ for all i and j. Set E_i the span of all elements in ε with $k \equiv i \pmod 2$. Then

* Corresponding author.
E-mail addresses: jfreitas@ime.unicamp.br (J.A. Freitas), plamen@ime.unicamp.br (P. Koshlukov).

[1] Supported by PhD grant from CAPES.

[2] Partially supported by CNPq Grant 302655/2005-0, and by FAPESP Grant 05/60337-2.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.09.031

E_0 is the centre of E and the elements of E_1 anticommute, and $E = E_0 \oplus E_1$ is $\mathbb{Z}_2$-graded. For short we call $\mathbb{Z}_2$-graded algebras simply 2-graded. The systematic study of 2-graded algebras was initiated by Wall in [17]. A turning point in the theory of graded algebras and their graded identities was achieved in Kemer's celebrated theory (see for an account [12]) that led to the classification of the T-ideals in characteristic 0, and to the positive solution of the long standing Specht problem. Let $K\langle X\rangle$ be the free associative algebra freely generated over K by the set X. If G is a finite abelian group we put $X = \bigcup_{g\in G} X_g$ where all X_g are infinite disjoint sets. The algebra $K\langle X\rangle$ is G-graded: $K\langle X\rangle_g$ is the span of all monomials $m = x_{i_1}x_{i_2}\dots x_{i_m}$ such that $x_{i_t} \in X_{g_t}$ and $g_1 + g_2 + \cdots + g_m = g$. The polynomial $f(x_1,\dots,x_n)$ is a G-graded identity for the G-graded algebra A if $f(a_1,\dots,a_n) = 0$ for every a_i such that the G-degree of a_i is the same as that of x_i. The set of all G-graded identities for A is the T_G-ideal $T_G(A)$. It is closed under the G-graded endomorphisms of A. Soon after Kemer's work graded identities became an object of independent interest. We refer to [5] for more information and recent results in graded PI theory.

According to Kemer's theory the T-prime algebras are fundamental in describing the T-ideals in characteristic 0. Recall that A is T-prime if its T-ideal is prime inside the class of the T-ideals in $K\langle X\rangle$. Such T-ideals are called T-prime as well. Kemer proved that the only T-prime T-ideals in characteristic 0 are 0, $K\langle X\rangle$, $T(M_n(K))$, $T(M_n(E))$, and $T(M_{k,l})$. Here $M_n(K)$ and $M_n(E)$ are the full matrix algebra over K and over E, respectively. Further $M_{k,l}$ is the subalgebra of $M_{k+l}(E)$ that consists of all matrices $\begin{pmatrix} u & v \\ w & t\end{pmatrix}$ where $u \in M_k(E_0)$, $t \in M_l(E_0)$, $v \in M_{k\times l}(E_1)$, $w \in M_{l\times k}(E_1)$. All these algebras are $\mathbb{Z}_2$-graded in a natural way, to be defined later on. Moreover Kemer proved that the tensor product of any two T-prime algebras is PI equivalent to a T-prime algebra.

Kemer's theorem. *Let* char $K = 0$. *Then*

$$M_{a,b} \otimes E \sim M_{a+b}(E); \qquad M_{a,b} \otimes M_{c,d} \sim M_{ac+bd,ad+bc}; \qquad M_{1,1} \sim E \otimes E.$$

The remaining tensor products of T-prime algebras from the list above yield isomorphisms.

It is known [1–3] that the tensor product theorem fails in characteristic $p > 2$ but it holds at multilinear level.

One of the first and most important results in the combinatorial PI theory was Regev's $A \otimes B$ theorem: the tensor product of two PI algebras is still a PI algebra. Regev's theorem has motivated a lot of research on PI algebras.

Another kind of a tensor product, the $\mathbb{Z}_2$-graded one, has long been used in physics, see [15] for its foundations and usage, and [8] for a formal approach. Assume A and B are $\mathbb{Z}_2$-graded, $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$. The $\mathbb{Z}_2$-graded tensor product $A \bar{\otimes} B$ of A and B is the vector space $A \otimes B$ with the multiplication $(a_1 \bar{\otimes} b_1)(a_2 \bar{\otimes} b_2) = (-1)^{|b_1||a_2|} a_1a_2 \bar{\otimes} b_1b_2$ for $a_i \in A_0 \cup A_1$, $b_i \in B_0 \cup B_1$. Here if $a \in A_0 \cup A_1$ is a homogeneous element we denote by $|a|$ its $\mathbb{Z}_2$-degree. Surprisingly enough the $\mathbb{Z}_2$-graded tensor product had not been considered in PI theory until the recent research of Regev and Seeman [14]. The authors of [14] proved the analog of the $A \otimes B$ theorem when one replaces the tensor product by the graded one. They introduced a powerful tool for studying the graded tensor product, the so-called regular decomposition, see for details [14]. Using this decomposition they studied the PI equivalence of graded tensor products, and conjectured that the graded tensor product of two T-prime algebras is PI equivalent to another T-prime algebra in characteristic 0. They proved several cases of such equivalences.

In this paper we deal with the remaining cases and thus confirm Regev and Seeman's conjecture. Moreover we show that in positive characteristic one cannot expect such a behaviour from the graded tensor product as in characteristic 0. More precisely we prove that whenever char $K = 0$ one has $T(M_{k,l} \bar{\otimes} E) = T(M_{k+l}(E))$ and $T(M_{k,l} \bar{\otimes} M_{r,s}) = T(M_{p,q})$ where $p = kr + ls$, $q = ks + lr$, exactly the same behaviour as in the case of ungraded tensor products. If char $K = p > 2$ we prove that $T(M_2(E)) \subset T(M_{1,1} \bar{\otimes} E)$, a proper inclusion. All these results are obtained by means of studying the graded identities satisfied by the corresponding algebras. When char $K = 0$ we apply several results from [9,10].

The $\mathbb{Z}_2$-graded tensor product can be generalised to the β-tensor product, also called *tensor product with commutation factors*. Let G be a finite additive abelian group, and let $\beta : G \times G \to K^*$ be a *skew-symmetric bicharacter*:

$$\beta(g+h,k)=\beta(g,k)\beta(h,k), \qquad \beta(g,h+k)=\beta(g,h)\beta(g,k), \qquad \beta(g,h)\beta(h,g)=1,$$

for every g, h, $k \in G$. It follows $\beta(g,0)=\beta(0,h)=1$, and $\beta(g,g)=\pm 1$ for every g, $h \in G$. Skew-symmetric bicharacters are most commonly used in the theory of colour Lie (super)algebras, see for example [5]. In [18] a description of the nondegenerate skew-symmetric bicharacters on a finite abelian group was given. The G-graded algebra A is β-commutative if $a_g a_h = (-1)^{\beta(g,h)} a_h a_g$ for every $a_g \in A_g$, $a_h \in A_h$. If we impose this condition on the free G-graded algebra $K(X)$ we get the free β-commutative algebra. In [18] it was obtained a description of the T-prime algebras (in characteristic 0) that are PI equivalent to a free β-commutative algebra.

If A and B are G-graded algebras, and β is a skew-symmetric bicharacter on G we define the β-tensor product $A \otimes_\beta B$ of A and B to be the vector space $A \otimes B$ equipped with the multiplication

$$(a_1 \otimes_\beta b_1)(a_2 \otimes_\beta b_2) = \beta(h_1, g_2) a_1 a_2 \otimes_\beta b_1 b_2, \quad a_i \in A_{g_i},\ b_i \in B_{h_i}. \tag{1}$$

This tensor product is well defined and associative, see [8]. If we set $C_k = \bigoplus_{g+h=k} A_g \otimes_\beta B_h$, $k \in G$, we have $A \otimes_\beta B = C = \bigoplus_{k\in G} C_k$ and C is G-graded. Identities in associative algebras with skew-symmetric bicharacter were considered by Zolotykh [18], and very recently by Berele [7].

In this paper we prove an analog of Regev's $A \otimes B$ theorem when one substitutes $\otimes$ by $\otimes_\beta$. In order to do that we first deduce an upper bound for the codimension sequence of $A \otimes_\beta B$ in terms of the codimensions of A and B. Moreover we prove that $M_n(K) \otimes_\beta E \sim M_n(E)$ where $M_n(K)$ is equipped with a fine $\mathbb{Z}_n \times \mathbb{Z}_n$-grading and E is $\mathbb{Z}_2$-graded as above. Recall that a grading on A by the group G is fine if $\dim A_g \leqslant 1$ for all $g \in G$.

1. The identities of $M_{k,l}(E) \bar{\otimes} E$ and $M_{k+l}(E)$

Here we assume that $\operatorname{char} K = 0$. The algebra $A = M_{k,l}$ is $\mathbb{Z}_2$-graded as follows: $A_0 = \{\left(\begin{smallmatrix} u & 0 \\ 0 & t\end{smallmatrix}\right) \mid u \in M_k(E_0), t \in M_l(E_0)\}$, $A_1 = \{\left(\begin{smallmatrix} 0 & v \\ w & 0\end{smallmatrix}\right) \mid v \in M_{k\times l}(E_1), w \in M_{l\times k}(E_1)\}$. Since $E = E_0 \oplus E_1$ is also 2-graded we have the graded tensor product $M_{k,l} \bar{\otimes} E$.

The algebra $M_n(K)$ is $\mathbb{Z}_n$-graded by setting $(M_n(K))_t = \operatorname{sp}\{e_{ij} \mid j - i \equiv t \pmod n\}$. The graded identities for this grading were described in [16].

If $k+l=n$ and $G = \mathbb{Z}_n \times \mathbb{Z}_2$ then the above $\mathbb{Z}_n$-grading on $M_n(K)$ induces a G-grading on $M_{k,l}$. If $(t,\lambda) \in G$ we put $(M_{k,l})_{(t,\lambda)}$ to be the span of all $E_\lambda e_{ij} \cap M_{k,l}$ such that $j - i \equiv t \pmod n$. Thus $M_{k,l} \bar{\otimes} E$ is G-graded by setting $(M_{k,l} \bar{\otimes} E)_{(t,\lambda)} = (M_{k,l})_{(t,\lambda)} \bar{\otimes} E_0 \oplus (M_{k,l})_{(t,\lambda+1)} \bar{\otimes} E_1$.

Define a function $\rho : \{1,\dots,n\} \times \{1,\dots,n\} \to \mathbb{Z}_2$ by $\rho(i,j) = 0$ whenever either $i \leqslant k$, $j \leqslant k$ or $i > k$, $j > k$, and set $\rho(i,j) = 1$ otherwise. Let $\varepsilon_i = \varepsilon \cap E_i$ be the natural basis of E_i, $i = 0, 1$. Then $\mathcal{A} = \{a e_{ij} \mid 1 \leqslant i, j \leqslant n, a \in \varepsilon_{\rho(i,j)}\}$ is a K-basis of $M_{k,l}$, and $\mathcal{B} = \{a e_{ij} \bar{\otimes} b \mid 1 \leqslant i, j \leqslant n, a \in \varepsilon_{\rho(i,j)}, b \in \varepsilon_\lambda, \lambda \in \mathbb{Z}_2\}$ is a K-basis of $M_{k,l}(E) \bar{\otimes} E$.

In order to simplify the notation we denote the elements $a \in \varepsilon_\lambda$ by a^λ, $\lambda \in \mathbb{Z}_2$. If an algebra is G-graded and its element a is homogeneous of G-degree (t,λ), we shall write $\partial(a) = (t,\lambda)$. If $t \in \mathbb{Z}$ then $\bar{t}$ is its residue modulo n, $\bar{t} \in \mathbb{Z}_n$. It follows that $(M_{k+l}(E))_{(t,\lambda)} = \operatorname{sp}\langle a^{\rho(i,j)} e_{ij} \mid \overline{j-i} = t, \rho(i,j) = \lambda\rangle$, and $(M_{k+l}(E) \bar{\otimes} E)_{(t,\lambda)} = \operatorname{sp}\{a^{\rho(i,j)} e_{ij} \bar{\otimes} b^{\lambda+\rho(i,j)} \mid \overline{j-i} = t, \rho(i,j) = \lambda\}$ for every $(t,\lambda) \in G$. All this makes the following lemma obvious.

Lemma 1. *The above defines a G-grading on the algebra $M_{k,l} \bar{\otimes} E$.*

Lemma 2. *Let $A_s = a_s^{\rho(i_s,j_s)} e_{i_s j_s} \bar{\otimes} b_s^{\lambda_s + \rho(i_s,j_s)} \in \mathcal{B}$, $s = 1, 2$, and let $A_1 A_2 \neq 0$. Then $c A_1 A_2 \in \mathcal{B}$ where $c = \pm 1$. Furthermore $j_1 = i_2$, $\rho(i_1,j_1) + \rho(i_2,j_2) = \rho(i_1,j_2)$, and $\partial(A_1 A_2) = (\overline{j_2 - i_1}, \lambda_1 + \lambda_2)$.*

Proof. Let $\rho(i_1, j_1) = \rho_1$ and $\rho(i_2, j_2) = \rho_2$. Since $A_1A_2 \neq 0$ one has $j_1 = i_2$. We prove $\rho(i_1, j_1) + \rho(i_2, j_2) = \rho(i_1, j_2)$ case by case. If, we say, $\rho_1 = \rho_2 = 1$ then necessarily the integer k is between i_1 and j_1. Therefore either $i_1 \leqslant k$ and $j_2 \leqslant k$ or $i_1 > k$, $j_2 > k$; in both cases $\rho(i_1, j_2) = 0 = \rho_1 + \rho_2$. The remaining three cases for ρ_1 and ρ_2 are dealt with analogously.

Finally let $a_1 = e_{l_1} \dots e_{l_r}$, $a_2 = e_{m_1} \dots e_{m_t}$, $l_1 < \dots < l_r$, $m_1 < \dots < m_t$. But $\{l_1, \dots, l_r\} \cap \{m_1, \dots, m_t\} = \emptyset$ since $A_1A_2 \neq 0$. Then we reorder, if necessary, the entries of a_1a_2 so that we obtain an element of $\varepsilon_{\rho_1+\rho_2}$. This element equals, up to a sign, a_1a_2. Analogously one deals with b_1b_2. Thus

$$0 \neq A_1A_2 = (-1)^{(\lambda_1+\rho(i_1,j_1))\rho(i_2,j_2)} a_1a_2 e_{i_1j_2} \bar{\otimes} b_1^{(\lambda_1+\rho(i_1,j_1))} b_2^{(\lambda_2+\rho(i_2,j_2))},$$

consequently $\partial(A_1A_2) = (\overline{j_2 - i_1}, \lambda_1 + \lambda_2)$. $\square$

Let $f = f(x_1, \dots, x_r) \in K\langle X\rangle$ be a multilinear polynomial and let $A_1, \dots, A_r \in \mathcal{B}$. The substitution $\mathcal{S}$ given by $x_i \mapsto A_i$, $i = 1, \dots, r$ is called *standard* if $\partial(x_i) = \partial(A_i)$ for all i; notation $\mathcal{S} : (A_1, \dots, A_r)$.

In characteristic 0, every graded identity for $M_{k,l} \bar{\otimes} E$ is equivalent to a finite collection of multilinear ones. A multilinear polynomial is a graded identity for $M_{k,l} \bar{\otimes} E$ if and only if it vanishes under all standard substitutions.

Lemma 3. *(See [16, Lemma 1].) Let $e_{i_1j_1}$, e_{ij}, $e_{i_2j_2} \in M_n(K)$ be matrix units such that $\overline{j_1 - i_1} = \overline{j_2 - i_2} = -\overline{j - i}$. Then $e_{i_1j_1}e_{ij}e_{i_2j_2} \neq 0$ if and only if $i_1 = j = i_2$ and $j_1 = i = j_2$. In this case $e_{i_1j_1}e_{ij}e_{i_2j_2} = e_{i_2j_2}e_{ij}e_{i_1j_1}$.*

Definition 1. Let $\mathcal{I}$ be the set of the G-graded multilinear polynomials

Identity	$(\alpha(x_1), \beta(x_1))$	$(\alpha(x_2), \beta(x_2))$	$(\alpha(x), \beta(x))$
$x_1x_2 - x_2x_1$	$(0,0)$	$(0,0)$	
	$(0,1)$	$(0,0)$	
$x_1x_2 + x_2x_1$	$(0,1)$	$(0,1)$	
$x_1xx_2 - x_2xx_1$	$(t,0)$	$(-t,0)$	$(t,0)$
	$(t,0)$	$(-t,1)$	$(t,0)$
	$(t,1)$	$(-t,0)$	$(t,0)$
$x_1xx_2 + x_2xx_1$	$(t,1)$	$(-t,1)$	$(t,0)$
	$(t,1)$	$(-t,0)$	$(t,1)$
	$(t,1)$	$(-t,1)$	$(t,1)$

$t \in \mathbb{Z}_n$, and let I be the ideal of G-graded identities in $K\langle X\rangle$ generated by $\mathcal{I}$.

Proposition 4. *(Cf. [10, Proposition 2.6].) $I \subseteq T_G(M_{k,l} \bar{\otimes} E)$.*

Proof. Since all polynomials in $\mathcal{I}$ are multilinear it suffices to show that these vanish under standard substitutions. But any standard substitution of degree 3 is of the form $A_1 = a_1^{\rho(i_1,j_1)} e_{i_1j_1} \bar{\otimes} b_1^{\lambda_1+\rho(i_1,j_1)}$, $A_2 = a_2^{\rho(i_2,j_2)} e_{i_2j_2} \bar{\otimes} b_2^{\lambda_2+\rho(i_2,j_2)}$, and $A = a^{\rho(i,j)} e_{ij} \bar{\otimes} b^{\lambda+\rho(i,j)}$. Here $\overline{j_1 - i_1} = \overline{j_2 - i_2} = \overline{i - j} = t \in \mathbb{Z}_n$. According to Lemma 3, both products $e_{i_1j_1}e_{ij}e_{i_2j_2}$ and $e_{i_2j_2}e_{ij}e_{i_1j_1}$ vanish unless $i_1 = i_2 = j$ and $j_1 = j_2 = i$. In the latter case they equal e_{ij}, and $\rho(i_1, j_1) = \rho(i_2, j_2) = \rho(i, j) = \rho$. Therefore the standard substitutions we have to consider are

$$A_1 = a_1^{\rho} e_{ij} \bar{\otimes} b_1^{\lambda_1+\rho}, \qquad A_2 = a_2^{\rho} e_{ji} \bar{\otimes} b_2^{\lambda_2+\rho}, \qquad A = a^{\rho} e_{ij} \bar{\otimes} b^{\lambda+\rho}$$

where $\overline{i - j} = t \in \mathbb{Z}_n$. Now we have the straightforward equalities

$$A_1AA_2 = (-1)^{\lambda\rho} a_1^{\rho} a^{\rho} a_2^{\rho} e_{ij} \bar{\otimes} b_1^{\lambda_1+\rho} b^{\lambda+\rho} b_2^{\lambda_2+\rho},$$
$$A_2AA_1 = (-1)^{\lambda\rho} a_2^{\rho} a^{\rho} a_1^{\rho} e_{ij} \bar{\otimes} b_2^{\lambda_2+\rho} b^{\lambda+\rho} b_1^{\lambda_1+\rho}.$$

So as in [10, Proposition 2.6] we conclude that the polynomials in $\mathcal{I}$ are G-graded identities for $M_{k,l} \bar{\otimes} E$. $\square$

Let $m = x_1x_2 \dots x_r$ be a multilinear monomial and let $\sigma \in S_r$ be a permutation. We denote by m_σ the monomial $x_{\sigma(1)} \dots x_{\sigma(r)}$. For any standard substitution $\mathcal{S}$ we denote by $m|_{\mathcal{S}}$ the evaluation of m on $\mathcal{S}$. By linearity these notations are extended to the case of multilinear polynomials in $x_1, \dots, x_r$.

Remark 5. Let m be a multilinear monomial of degree r and let $\sigma \in S_r$. It is an easy exercise to find a standard substitution $\mathcal{S}$ such that $m|_{\mathcal{S}} \neq 0$.

Definition 2. Let $m = x_1 \dots x_r$ be a multilinear monomial and let $\sigma \in S_r$. If $1 \leqslant p \leqslant q \leqslant r$ we define $m_\sigma^{[p,q]} = x_{\sigma(p)} \dots x_{\sigma(q)}$.

Remark 6. Let $\mathcal{S} : (A_1, \dots, A_r)$ be a standard substitution. For $s = 1, \dots, r$, fix $x_s \mapsto A_s = a_s^{\rho(i_s,j_s)} e_{i_s j_s} \bar{\otimes} b_s^{\lambda_s+\rho(i_s,j_s)}$. Here $\partial(x_s) = (\overline{j_s - i_s}, \lambda_s) = \partial(A_s)$.

Let $m_\sigma|_{\mathcal{S}} = A_{\sigma(1)} \dots A_{\sigma(r)} \neq 0$. Then $m_\sigma|_{\mathcal{S}} = cA$ for some $A \in \mathcal{B}$ and $c \in \{-1, 1\}$, by Lemma 2. Moreover $m_\sigma|_{\mathcal{S}} \neq 0$ if and only if $m_\sigma^{[p,q]}|_{\mathcal{S}} \neq 0$ for every $1 \leqslant p \leqslant q \leqslant r$, and in this situation $\partial(m_\sigma^{[p,q]}) = (\overline{j_{\sigma(q)} - i_{\sigma(p)}}, \lambda_{\sigma(p)} + \dots + \lambda_{\sigma(q)})$.

We observe that again by Lemma 2 it holds

$$\begin{aligned}
\partial\big(m_\sigma^{[p,q]}\big) &= \partial(x_{\sigma(p)}) + \dots + \partial(x_{\sigma(q)}) \\
&= (\overline{j_{\sigma(p)} - i_{\sigma(p)}}, \lambda_{\sigma(p)}) + \dots + (\overline{j_{\sigma(q)} - i_{\sigma(q)}}, \lambda_{\sigma(q)}) \\
&= (\overline{j_{\sigma(q)} - i_{\sigma(p)}}, \lambda_{\sigma(p)} + \dots + \lambda_{\sigma(q)}).
\end{aligned}$$

The next couple of statements as well as their proofs are quite similar to [10, Lemmas 3.1, 3.2].

Lemma 7. *Let $\mathcal{S}$ be a standard substitution such that $m_\sigma|_{\mathcal{S}} = \pm m|_{\mathcal{S}} \neq 0$. Then there exists $c \in \{-1, 1\}$ with $m_\sigma \equiv cx_1m'(x_2, \dots, x_r) \pmod{I}$.*

Lemma 8. *If $\mathcal{S}$ is a standard substitution such that $m_\sigma|_{\mathcal{S}} = cm|_{\mathcal{S}} \neq 0$ for some $c \in \{-1, 1\}$ then $m_\sigma \equiv cm'(x_2, \dots, x_r) \pmod{I}$.*

Corollary 9. *Let $\sigma, \tau \in S_r$. Assume that for some standard substitution $\mathcal{S}$ we have $m_\sigma|_{\mathcal{S}} = cm_\tau|_{\mathcal{S}} \neq 0$ where $c \in \{-1, 1\}$. Then $m_\sigma \equiv cm_\tau \pmod{I}$.*

Theorem 10. *Let $n = k + l$. The set $\mathcal{I}$ from Definition 1 generates the ideal of G-graded identities $T_G(M_{k,l} \bar{\otimes} E)$. In other words $I = T_G(M_{k,l} \bar{\otimes} E)$.*

Proof. Proposition 4 gives us one of the inclusions. So it suffices to show that every multilinear graded identity for $M_{k,l} \bar{\otimes} E$ lies in I.

Suppose on the contrary that $f = f(x_1, \dots, x_r)$ is a G-graded multilinear polynomial such that $f \in T_G(M_{k,l} \bar{\otimes} E) \setminus I$. Write $f \equiv \sum_{s=1}^t d_{\sigma_s} m_{\sigma_s} \pmod{I}$ for some multilinear monomials m_{σ_s}, $\sigma_s \in S_r$, and nonzero scalars $d_{\sigma_s} \in K$, $s = 1, \dots, t$. Among all f with these properties take one with the least possible t. Notice that by Remark 6 we must have $t \geqslant 2$. Moreover there is a standard substitution $\mathcal{S}$ such that $m_{\sigma_1}|_{\mathcal{S}} \neq 0$. But f vanishes on $M_{k,l} \bar{\otimes} E$, in particular $f|_{\mathcal{S}} = 0$. Therefore $d_{\sigma_1} m_{\sigma_1}|_{\mathcal{S}} = -\sum_{s=2}^t d_{\sigma_s} m_{\sigma_s}|_{\mathcal{S}}$.

By Remark 5, $0 \neq m_{\sigma_1}|_{\mathcal{S}} = c_1 A$ for some $A \in \mathcal{B}$ and $c_1 \in \{-1, 1\}$. Thus for some p, $1 \leqslant p \leqslant t$, it holds $0 \neq m_{\sigma_p}|_{\mathcal{S}} = c_2 A$ where $c_2 \in \{-1, 1\}$. In this way $0 \neq m_{\sigma_1}|_{\mathcal{S}} = c_1 c_2^{-1} m_{\sigma_p}|_{\mathcal{S}}$, and by Corollary 9, $m_{\sigma_p} \equiv c m_{\sigma_1} \pmod{I}$ where $c = c_1 c_2^{-1}$. Therefore $f \equiv (d_{\sigma_1} + c d_{\sigma_p}) m_{\sigma_1} + \sum_s d_{\sigma_s} m_{\sigma_s} \pmod{I}$, the second sum runs over $s = 2, \ldots, t$, $s \neq p$. This contradicts the minimality of t. $\square$

One of the central results, Theorem 4.1, in [10] is the following.

Theorem 11. *The set $\mathcal{I}$ generates the ideal of G-graded identities for $M_n(E)$.*

Corollary 12. *If $k + l = n$ then $T_G(M_{k,l} \bar{\otimes} E) = T_G(M_n(E))$.*

Since two G-graded algebras satisfying the same G-graded identities are PI equivalent, we obtain one of the cases of the *graded* tensor product theorem.

Theorem 13. *If $k + l = n$ then $T(M_{k,l} \bar{\otimes} E) = T(M_n(E))$.*

2. The identities of $M_{p,q} \bar{\otimes} M_{r,s}$

The algebras $M_{p,q}$ and $M_{r,s}$ are $\mathbb{Z}_2$-graded as in the preceding section. Thus we have the graded tensor product $M_{p,q} \bar{\otimes} M_{r,s}$ with respect to these gradings.

If $m = p + q$ and $n = r + s$ we define the functions $\alpha : \{1, \ldots, m\} \to \mathbb{Z}_2$ and $\beta : \{1, \ldots, n\} \to \mathbb{Z}_2$ as follows. Set $\alpha(x) = 0$ whenever $1 \leqslant x \leqslant p$ and $\alpha(x) = 1$ otherwise; also $\beta(x) = 0$ if $1 \leqslant x \leqslant r$, and $\beta(x) = 1$ otherwise. Let G be an abelian group and let R be a G-graded algebra, $R = \bigoplus_{g \in G} R_g$. Assume that $\mathcal{B}$ is a K-basis of R and that the elements of $\mathcal{B}$ are G-homogeneous. As in [9] we call $\mathcal{B}$ a *standard basis* of R if for every $a, b \in \mathcal{B}$ such that $ab \neq 0$ there exists $\rho \in K$ such that $\rho ab \in \mathcal{B}$.

Here we modify the definition of a standard substitution as follows. Let $\mathcal{S} : K(X) \to R$ be a homomorphism of G-graded algebras such that $\mathcal{S}(x_i) = a_i \in \mathcal{B}$ for every $x_i \in X$. Then $\mathcal{S}$ is a *standard substitution*. If $f \in K(X)$ is a multilinear polynomial we denote by $f|_{\mathcal{S}}$ the image of f under $\mathcal{S}$.

The natural basis ε of the Grassmann algebra E is standard with $\rho = \pm 1$. It was shown in [9, Section 2] that the following set is a basis of $M_{p,q} \otimes M_{r,s}$

$$\mathcal{B} = \{ a e_{ij} \otimes b e_{uv} \mid a \in \varepsilon_{\alpha(j)-\alpha(i)},\ b \in \varepsilon_{\beta(v)-\beta(u)};\ 1 \leqslant i, j \leqslant m;\ 1 \leqslant u, v \leqslant n \}.$$

If $G = \mathbb{Z}_{mn} \times \mathbb{Z}_2$ then $M_{p,q} \otimes M_{r,s}$ is G-graded by setting

$$\partial_G(a e_{ij} \otimes b e_{uv}) = \bigl(\overline{n(j-i) + v - u}, \alpha(j) + \beta(v) - \bigl(\alpha(i) - \alpha(u)\bigr) \bigr) \in G$$

for each $a e_{ij} \otimes b e_{uv}$ from the basis $\mathcal{B}$ of $M_{p,q} \otimes M_{r,s}$. The basis $\mathcal{B}$ is G-homogeneous; it is standard, see [9]. Hence the G-graded algebra $M_{p,q} \otimes M_{r,s}$ has a standard basis, namely $\mathcal{B}$.

The vector spaces $M_{p,q} \otimes M_{r,s}$ and $M_{p,q} \bar{\otimes} M_{r,s}$ coincide therefore $\mathcal{B}$ is a basis of the latter algebra. Since $M_{p,q} \bar{\otimes} M_{r,s}$ is G-graded in the same way as $M_{p,q} \otimes M_{r,s}$ its basis $\mathcal{B}$ is G-homogeneous and it is also a standard basis for $M_{p,q} \bar{\otimes} M_{r,s}$. Let $\mathcal{M}$ be the set of all multilinear monomials in the free G-graded algebra $K(X)$.

Proposition 14. *(See [9, Proposition 8].) Let R be a G-graded algebra with a standard basis $\mathcal{B}$. Further let the set $\mathcal{N}$ of G-graded identities of R generate the T_G-ideal I. Assume for each m and $m' \in \mathcal{M} \setminus T_G(R)$ there exists a standard substitution $\mathcal{S}$ and a scalar $0 \neq c \in K$ such that $m|_{\mathcal{S}} = c m'|_{\mathcal{S}}$ if and only if $m \equiv c m' \pmod{\mathcal{I}}$. Then the set $\mathcal{N} \cup (\mathcal{M} \cap T_G(R))$ generates $T_G(R)$.*

We fix the algebra $R = M_{p,q} \bar{\otimes} M_{r,s}$ graded by the group $G = \mathbb{Z}_{mn} \times \mathbb{Z}_2$ as described above. Let $0 \neq w = a e_{ij} \bar{\otimes} b e_{uv} \in R_{(0,\delta)}$ then $n(j-i) + v - u = kmn$ for some integer k. But n divides $v - u$ and

$1 \leqslant u, v \leqslant n$ hence $u = v$. Similarly $i = j$ thus $\delta = 0$ and $a, b \in E_{\alpha(j)-\alpha(i)} = E_{\beta(j)-\beta(i)} = E_0$. Therefore $R_{(0,1)} = 0$ and every monomial in $K(X)$ of G-degree $(0,1)$ is a graded identity for R.

We denote by $\mathcal{N}$ the following collection of graded multilinear polynomials:

$$x_1x_2 - x_2x_1 \quad \text{where } \partial_G(x_1) = \partial_G(x_2) = (0,0),$$

$$x_1x_2x_3 - x_3x_2x_1 \quad \text{where } \partial_G(x_1) = \partial_G(x_3) = -\partial_G(x_2) = (t,0),$$

$$x_1x_2x_3 + x_3x_2x_1 \quad \text{where } \partial_G(x_1) = \partial_G(x_3) = -\partial_G(x_2) = (t,1).$$

Lemma 15. $\mathcal{N} \subseteq T_G(R)$.

Proof. If w is a G-homogeneous element of a G-graded algebra we denote by $\gamma(w) \in \mathbb{Z}_{mn}$ and $\delta(w) \in \mathbb{Z}_2$ its $\mathbb{Z}_{mn}$ and $\mathbb{Z}_2$-degrees, respectively.

Let $w = a_h e_{i_h j_h} \bar{\otimes} b_h e_{u_h v_h} \in \mathcal{B}$, $h = 1, 2, 3$ and let $\gamma(w_1) = \gamma(w_3) = -\gamma(w_2)$. Suppose $w_1w_2w_3 \neq 0$ then $w_1w_2 \neq 0$ and $\delta(w_1w_2) = 0$. But $w_1w_2 \neq 0$ implies $j_1 = i_2$ and $v_1 = u_2$. Furthermore $j_2 = i_1$ and $u_1 = v_2$. In the same manner one gets $j_2 = i_3$, $v_2 = u_3$, $j_3 = i_2$ and $v_3 = u_2$. Thus $w_1 = a_1 e_{ij} \bar{\otimes} b_1 e_{uv}$, $w_2 = a_2 e_{ji} \bar{\otimes} b_2 e_{vu}$ and $w_3 = a_3 e_{ij} \bar{\otimes} b_3 e_{uv}$ where $1 \leqslant i, j \leqslant m$ and $1 \leqslant u, v \leqslant n$. The following equalities hold:

$$w_1w_2w_3 = (-1)^{\partial_{\mathbb{Z}_2}(b_1e_{uv})\partial_{\mathbb{Z}_2}(a_2e_{ji})} a_1a_2a_3e_{ij} \bar{\otimes} b_1b_2b_2e_{uv},$$

$$w_3w_2w_1 = (-1)^{\partial_{\mathbb{Z}_2}(b_3e_{uv})\partial_{\mathbb{Z}_2}(a_2e_{ji})} a_3a_2a_1e_{ij} \bar{\otimes} b_3b_2b_1e_{uv}.$$

Since $b_1, b_3 \in E_{\beta(v)-\beta(u)}$ we obtain $\partial_{\mathbb{Z}_2}(b_1e_{uv}) = \partial_{\mathbb{Z}_2}(b_3e_{uv})$ and consequently $(-1)^{\partial_{\mathbb{Z}_2}(b_1e_{uv})\partial_{\mathbb{Z}_2}(a_2e_{ji})} = (-1)^{\partial_{\mathbb{Z}_2}(b_3e_{uv})\partial_{\mathbb{Z}_2}(a_2e_{ji})}$. Finally for $h = 1, 2, 3$ we have $a_h \in E_{\alpha(j)-\alpha(i)} = E_{\alpha(i)-\alpha(j)}$ and $b_h \in E_{\beta(v)-\beta(u)} = E_{\beta(u)-\beta(v)}$. $\square$

Lemma 16. *Let m and m' be multilinear monomials on the same collection of variables. Suppose that for some standard substitution $\mathcal{S}$ in $R = M_{p,q} \bar{\otimes} M_{r,s}$ it holds $m'|_{\mathcal{S}} = cm|_{\mathcal{S}} \neq 0$ for some $c \in K$. Then $m' \equiv cm \pmod I$.*

Proof. Suppose $m = x_1 \dots x_d$ then $m' = m_\sigma = x_{\sigma(1)} \dots x_{\sigma(d)}$ for some $\sigma \in S_d$. We induct on d, the base $d = 1$ being trivial. Let $d \geqslant 2$. First we show that there exists $c' \in K$ with $m' \equiv c'x_1m''(x_2, \dots, x_d) \pmod I$. If $1 \leqslant a \leqslant b \leqslant d$ we denote $w^{[a,b]} = x_a \dots x_b$. Let $\mathcal{S}(x_h) = w_h = a_h e_{i_h j_h} \bar{\otimes} b_h e_{u_h v_h} \in \mathcal{B}$, $h = 1, \dots, d$. As $m'|_{\mathcal{S}} = cm|_{\mathcal{S}} \neq 0$ we have $i_1 = i_{\sigma(1)}$ and $u_1 = u_{\sigma(1)}$.

If $\sigma^{-1}(1) = 1$ we apply the induction. Thus suppose $\sigma^{-1}(1) > 1$ and denote $t = \min\{j \geqslant d \mid 1 \leqslant \sigma^{-1}(j) < \sigma^{-1}(1)\}$. We have $t > 1$ and $1 \leqslant \sigma^{-1}(t) < \sigma^{-1}(1) \leqslant \sigma^{-1}(t-1)$. Set $l = \sigma^{-1}(t)$, $k = \sigma^{-1}(t-1)$, $h = \sigma^{-1}(1)$; now consider the two possibilities: $l = 1$ and $l > 1$. The case $l = 1$ yields

$$\gamma\left(m_\sigma^{[1,h-1]}\right) = \gamma(x_{\sigma(1)} \dots x_{\sigma(h-1)}) = \overline{(r+s)(j_{\sigma(k-1)} - i_{\sigma(1)}) + v_{\sigma(h-1)} - u_{\sigma(1)}},$$
$$\gamma\left(m_\sigma^{[h,k]}\right) = \gamma(x_{\sigma(h)} \dots x_{\sigma(k)}) = \overline{(r+s)(j_{\sigma(k)} - i_{\sigma(k)}) + v_{\sigma(k)} - u_{\sigma(h)}}.$$

Since $\mathcal{S}$ vanishes neither $m_\sigma^{[1,h-1]}$ nor $m_\sigma^{[h,k]}$ we obtain

$$j_{\sigma(h-1)} - i_{\sigma(1)} = i_{\sigma(h)} - i_1 = i_1 - i_1 = 0,$$

$$v_{\sigma(h-1)} - u_{\sigma(1)} = u_{\sigma(h)} - u_1 = u_1 - u_1 = 0,$$

$$j_{\sigma(k)} - i_{\sigma(h)} = j_{t-1} - i_1 = i_t - i_1 = i_{\sigma(l)} - i_1 = i_{\sigma(1)} - i_1 = i_1 - i_1 = 0,$$

$$v_{\sigma(k)} - u_{\sigma(h)} = v_{t-1} - u_1 = u_t - u_1 = u_{\sigma(l)} - u_1 = u_{\sigma(1)} - u_1 = u_1 - u_1 = 0.$$

Hence $\gamma(m_\sigma^{[1,h-1]}) = \gamma(m_\sigma^{[h,k]}) = 0$. Moreover $\delta(m_\sigma^{[1,h-1]}) = \delta(m_\sigma^{[h,k]}) = 0$, since were it on the contrary the monomials would have been graded identities for R due to $R_{(0,1)} = 0$.

Since $x_1x_2 - x_2x_1 \in I$ whenever $\partial_G(x_1) = \partial_G(x_2) = (0,0)$ we have that $m' \equiv m_\sigma^{[h,k]} m_\sigma^{[1,h-1]} m_\sigma^{[k+1,d]}$ and $m_\sigma^{[h,k]}$ starts with x_1.

When $l > 1$ we obtain similarly $\gamma(m_\sigma^{[1,l-1]}) = -\gamma(m_\sigma^{[l,h-1]}) = \gamma(m_\sigma^{[h,k]})$ and $\delta(m_\sigma^{[1,l-1]}) = -\delta(m_\sigma^{[l,h-1]}) = \delta(m_\sigma^{[h,k]})$. Since $\mathcal{N} \subseteq I$ there exists $c \in \{-1,1\}$ with $m' \equiv c' m_\sigma^{[h,k]} m_\sigma^{[l,h-1]} m_\sigma^{[k+1,d]} \pmod I$ therefore $m_\sigma^{[h,k]}$ begins with x_1.

In both cases above we obtain that $c'x_1m''(x_2,\dots,x_d) \equiv m' \pmod I$, thus $c'x_1m''(x_2,\dots,x_d)|_{\mathcal{S}} = cm|_{\mathcal{S}}$. Hence $m''(x_2,\dots,x_d)|_{\mathcal{S}} = (c')^{-1}cx_2\dots x_d|_{\mathcal{S}}$. By the inductive assumption we get $m''(x_2,\dots,x_d) \equiv c''x_2\dots x_d \pmod I$ where $c'' = (c')^{-1}c$. In this way $m' \equiv c''x_1\dots x_d \pmod I$. $\square$

Putting together Lemma 16 and Proposition 14 we have

Theorem 17. *$T_G(M_{p,q} \bar{\otimes} M_{r,s})$ is generated by $\mathcal{N} \cup (\mathcal{M} \cap T_G(R))$, that is by the polynomials*

$$x_1x_2 - x_2x_1 \quad \text{with } \partial_G(x_1) = \partial_G(x_2) = (0,0),$$

$$x_1x_2x_3 - x_3x_2x_1 \quad \text{with } \partial_G(x_1) = \partial_G(x_3) = -\partial_G(x_2) = (t,0),$$

$$x_1x_2x_3 + x_3x_2x_1 \quad \text{with } \partial_G(x_1) = \partial_G(x_3) = -\partial_G(x_2) = (t,1),$$

and by all multilinear monomials that are graded identities for $M_{p,q} \bar{\otimes} M_{r,s}$.

Now we have one more case of the graded tensor product theorem.

Theorem 18. *When* $\operatorname{char} K = 0$ *the algebras $M_{p,q} \bar{\otimes} M_{r,s}$ and $M_{pr+qs,ps+qr}$ are PI equivalent.*

Proof. According to [9, Theorem 12], the set from Theorem 17 generates the ideal of G-graded identities of $M_{pr+qs,ps+qr}$, thus $T_G(M_{p,q} \bar{\otimes} M_{r,s}) = T_G(M_{pr+qs,ps+qr})$, and consequently $T(M_{p,q} \bar{\otimes} M_{r,s}) = T(M_{pr+qs,ps+qr})$. $\square$

3. $M_{1,1} \bar{\otimes} E$ and $M_2(E)$ in characteristic p: a negative result

We fix an infinite field K, $\operatorname{char} K = p > 2$. Let $X = \bigcup_{(i,j)\in\{0,1\}} X_{(i,j)}$ where $X_{(0,0)} = \{u_i\}$, $X_{(0,1)} = \{v_i\}$, $X_{(1,0)} = \{t_i\}$ and $X_{(1,1)} = \{w_i\}$, $i = 1, 2, \dots$, infinite disjoint sets of variables. Set $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. The algebra $K\langle X\rangle$ is then G-graded in a natural way. On the other hand put $Y = \{u_i, w_i\}$ and $Z = \{v_i, t_i\}$. One defines a $\mathbb{Z}_2$-grading in $K\langle X\rangle$ by letting Y be even and Z odd variables. Let T be the ideal in $K\langle X\rangle$ generated by all $ab - (-1)^{\partial(a)\partial(b)}ba$ where a and b are homogeneous monomials. Define $\Omega = K\langle X\rangle/T$. Then Ω is the free supercommutative algebra, and moreover $\Omega \cong K[Y] \otimes_K E(Z)$. Here $K[Y]$ is the commutative polynomial algebra on Y, and $E(Z)$ is the Grassmann algebra of the vector space with a basis Z, see [6, Section 2].

Now let $Y = \{y_{ij}^k\}$, $Z = \{z_{ij}^k\}$, and let $\Omega = K[Y] \otimes E(Z)$ be the corresponding free supercommutative algebra. Following [6] we construct the relatively free algebra in the variety of algebras generated by $M_2(E)$. This relatively free algebra is a subalgebra of $M_2(\Omega)$. Let $B^{(k)} \in M_2(\Omega)$ be the matrix whose (i,j)th entry is $(y_{ij}^k + z_{ij}^k)$, and let $C^{(k)} = \begin{pmatrix} y_{11}^k & z_{12}^k \\ z_{21}^k & y_{22}^k \end{pmatrix} \in M_2(\Omega)$.

Theorem 19. *(See [6, Theorem 2].)*

(1) *The matrices $B^{(1)}, B^{(2)}, \dots$, generate a subalgebra of $M_2(\Omega)$ that is relatively free in* $\operatorname{var} M_2(E)$.
(2) *The matrices $C^{(1)}, C^{(2)}, \dots$, generate a subalgebra of $M_2(\Omega)$ that is relatively free in* $\operatorname{var} M_{1,1}$.

Let $A^{(i)} = \begin{pmatrix} u_i & 0 \\ 0 & w_i \end{pmatrix}$, $D^{(i)} = \begin{pmatrix} 0 & v_i \\ t_i & 0 \end{pmatrix}$ where $u_i, w_i \in \Omega_0$ and $t_i, v_i \in \Omega_1$. Take another copy SC of the free supercommutative algebra and suppose SC freely generated by the even variables $\{y_i\}$ and by the odd ones $\{z_i\}$. If $r, s \in \mathbb{Z}_2$ we define $C_i^{(r,s)} \in M_2(\Omega) \bar{\otimes} SC$ as follows:

$$C_i^{(0,0)} = A^{(i)} \bar{\otimes} y_i, \qquad C_i^{(0,1)} = A^{(i)} \bar{\otimes} z_i, \qquad C_i^{(1,0)} = D^{(i)} \bar{\otimes} y_i, \qquad C_i^{(1,1)} = D^{(i)} \bar{\otimes} z_i,$$

where $i = 1, 2, \ldots$. The matrices $C_i^{(r,s)}$, $r, s \in \mathbb{Z}_2$, $i \geqslant 1$, generate an algebra F. This algebra is G-graded in a natural way.

Let $\psi : K(X) \to F$ be the algebra homomorphism defined by $u_i \mapsto C_i^{(0,0)}$, $v_i \mapsto C_i^{(0,1)}$, $t_i \mapsto C_i^{(1,0)}$, $w_i \mapsto C_i^{(1,1)}$. Then ψ is G-graded, epi, and moreover $\ker \psi = T_G(M_{1,1} \bar{\otimes} E)$. This means that $F \cong K(X)/T_G(M_{1,1} \bar{\otimes} E)$. We consider $P = M_{1,1} \bar{\otimes} E$ with the G-grading defined at the beginning of Section 1. Namely if $M_{1,1} = M_0 \oplus M_1$ is the usual 2-grading on $M_{1,1}$ then

$$P_{(0,0)} = M_0 \bar{\otimes} E_0, \qquad P_{(0,1)} = M_0 \bar{\otimes} E_1, \qquad P_{(1,0)} = M_1 \bar{\otimes} E_1, \qquad P_{(1,1)} = M_1 \bar{\otimes} E_0.$$

Denote by S the following collection of graded identities.

Identity	$(\alpha(x_1), \beta(x_1))$	$(\alpha(x_2), \beta(x_2))$	$(\alpha(x_3), \beta(x_3))$
$x_1x_2 - x_2x_1$	$(0,0)$	$(0,0)$	
	$(0,1)$	$(0,0)$	
$x_1x_2 + x_2x_1$	$(0,1)$	$(0,1)$	
$x_1x_2x_3 - x_3x_2x_1$	$(t,0)$	$(t,0)$	$(t,1)$
	$(t,0)$	$(t,1)$	$(t,0)$
	$(t,1)$	$(t,0)$	$(t,0)$
$x_1x_2x_3 + x_3x_2x_1$	$(t,0)$	$(t,1)$	$(t,1)$
	$(t,1)$	$(t,0)$	$(t,1)$
	$(t,1)$	$(t,1)$	$(t,1)$

where $t \in \mathbb{Z}_2$. Let I' be the ideal of G-graded identities generated by S and by all monomials in $T_G(P)$. The next couple of statements are proved exactly in the same way as their counterparts in [3].

Lemma 20. *$I' \subseteq T_G(P)$.*

Corollary 21. *$T_G(M_2(E)) \subseteq T_G(M_{1,1} \bar{\otimes} E)$, and $T(M_2(E)) \subseteq T(M_{1,1} \bar{\otimes} E)$.*

Let $M(x_1, \ldots, x_m)$, $N(x_1, \ldots, x_m) \in K(X)$ be monomials. Denote by F' the algebra generated by the matrices $A^{(i)}$ and $D^{(i)}$, $i \geqslant 1$. It is G-graded in an obvious manner. Let $\varphi : K(X) \to F'$ be the G-graded homomorphism defined by $\varphi(x_i) = A^{(i)}$ or $D^{(i)}$ depending on $\partial(x_i)$.

Proposition 22. *(Cf. [3, Proposition 14].) If the matrices $\varphi(M(x_1, \ldots, x_m))$ and $\pm\varphi(N(x_1, \ldots, x_m))$ share at some position the same nonzero entry then $M \equiv \pm N \pmod{I'}$.*

Theorem 23. *The G-graded identities of $M_{1,1} \bar{\otimes} E$ are consequences from the polynomials in S and from all monomials in $T_G(M_{1,1} \bar{\otimes} E)$.*

Proof. Let $f \in T_G(M_{1,1} \bar{\otimes} E)$ be multihomogeneous, it suffices to show that $f \in I'$. Let $r \geqslant 0$ be the least integer with $f \equiv \sum_{i=1}^r a_i f_i \pmod{I'}$ where f_i are monomials in $K(X)$, and $a_i \neq 0$. Supposing $r > 0$ we have $f_1 \notin T_G(M_{1,1} \bar{\otimes} E)$ due to the choice of r. Then for some $\delta_k = \gamma_{t_{k,1}} + \cdots + \gamma_{t_{k,n_k}}$, $t_{r,s} \neq t_{u,v}$ when $(u,v) \neq (r,s)$ we will have $f_1(\delta_1, \ldots, \delta_m) \neq 0$. Let $f_1 = x_{i_1} \ldots x_{i_q}$. Put $J = \{t_{i,j} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n_i\}$. We define a function $g : J \to \{1, \ldots, m\}$ by $g(t_{u,v}) = u$. As

$f_1(\delta_1,\dots,\delta_m)\neq 0$ there is a term $H_{j_1}\dots H_{j_q}\bar{\otimes} e\neq 0$ which does not cancel out. (Here $e\in SC$.) We notice that $g(j_k)=i_k$.

Therefore $a_1f_1(\delta_1,\dots,\delta_m)=-\sum_{i=2}^r a_if_i(\delta_1,\dots,\delta_m)$, and the right-hand side contains a term $H_{j_{\sigma(1)}}\dots H_{j_{\sigma(q)}}\bar{\otimes} e$, for some $\sigma\in S_q$. But $H_{j_1}\dots H_{j_q}$ and $H_{j_{\sigma(1)}}\dots H_{j_{\sigma(q)}}$ share the same nonzero entry in some position. If the last term above comes from $f_2=x_{s_1}\dots x_{s_q}$ then $g(j_{\sigma(k)})=s_k$. Now let $h_1(x_1,\dots,x_q)=x_{j_1}\dots x_{j_q}$ and $h_2(x_1,\dots,x_q)=x_{j_{\sigma(1)}}\dots x_{j_{\sigma(q)}}$. Applying Proposition 22 we get $h_1\equiv\pm h_2 \pmod{I'}$. Since $i_{\sigma(k)}=g(j_{\sigma(k)})=s_k$ we have $f_1(x_1,\dots,x_m)=h_1(x_{j_1},\dots,x_{j_q})$ and $f_2(x_1,\dots,x_m)=h_2(x_{j_1},\dots,x_{j_q})$. Hence $f_1\equiv\pm f_2 \pmod{I'}$ and $f\equiv(a_1\pm a_2)f_1+\sum_{i=3}^r a_if_i \pmod{I'}$. But this contradicts the choice of r. □

We proceed in a way similar to that of [3, Section 5]. Let $c(x_1,\dots,x_p)=x_1x_2x_1x_3x_1\dots x_1x_px_1$ where $\alpha(x_i)=1$ for each i and $\beta(x_1)=0$. Denote by I_1 the ideal of G-graded identities generated by I' and $c(x_1,\dots,x_p)$.

Lemma 24. *(Cf. [3, Lemma 18].) If* $\operatorname{char} K=p>2$ *then* $I_1\subseteq T_G(M_{1,1}\bar{\otimes} E)$.

Proof. It suffices to show that $c(x_1,\dots,x_p)\in T_G(M_{1,1}\bar{\otimes} E)$. Since c is linear in $x_2,\dots,x_p$ we can substitute x_1 by $a_j=\sum_{j=1}^r\left(\begin{smallmatrix}0&\alpha_j\\ \beta_j&0\end{smallmatrix}\right)\bar{\otimes}\gamma_j$ and x_i, $i>1$, by $a_i=\left(\begin{smallmatrix}0&\delta_i\\ \rho_i&0\end{smallmatrix}\right)\bar{\otimes}\varphi_i$. Here $\alpha_j,\ \beta_j,\ \gamma_j,\ \delta_i,\ \rho_i\in E_1$ and $\varphi_i\in E_0\cup E_1$. For $i>1$ we have $a_1a_i=-\sum_{j=1}^r\left(\begin{smallmatrix}\alpha_j\rho_i&0\\ 0&\beta_j\delta_i\end{smallmatrix}\right)\bar{\otimes}\gamma_j\varphi_i$. Since $t^2=0$ for each $t\in E_1$ we get $c(x_1,\dots,x_p)=0$ whenever $r<p$. If $r=p$ then $c(x_1,\dots,x_p)=-\sum_{\sigma\in S_p}\left(\begin{smallmatrix}0&u_\sigma\\ v_\sigma&0\end{smallmatrix}\right)\bar{\otimes} w_\sigma$. Here $u_\sigma=\alpha_{\sigma(1)}\rho_2\alpha_{\sigma(2)}\rho_3\dots\alpha_{\sigma(p-1)}\rho_p\alpha_{\sigma(p)}$, and similar expressions for v_σ, w_σ. Thus $c(a_1,\dots,a_p)=-(p!)\left(\begin{smallmatrix}0&u_1\\ v_1&0\end{smallmatrix}\right)\bar{\otimes} w_1=0$ in K since α_i, β_i, γ_i and ρ_i anticommute. The case $r>p$ is reduced to the previous ones by taking subsets of p elements each. □

Remark 25. As it was done in [3] we see that $c(x_1,\dots,x_p)\notin T_G(M_2(E)\bar{\otimes} E)$. Therefore $T_G(M_2(E))\subset T_G(M_{1,1}\bar{\otimes} E)$, a proper inclusion.

Lemma 26.

(1) *If $f\in K(X)$ is a monomial, and $x\in X$ is such that $\beta(x)=1$ and $\alpha(x)=\alpha(f)$ then $xfx\in I_1$.*
(2) *If $f_1,\ f_2\in K(X)$ are monomials, and $x\in X$ is such that $\beta(x)=1$ then $xf_1xf_2x\in I_1$.*

Proof. (1) When $\alpha(x)=1$ by $x_1x_2x_3+x_3x_2x_1=0$ we have $xfx\equiv -xfx \pmod{I_1}$, hence $xfx\in I_1$. If $\alpha(x)=0$ we can assume $\deg f\geqslant 1$ since for $\deg f=1$ we use $x_1x_2=-x_2x_1$ and $x^2\in I_1$. When $\beta(f)=0$ we use $x_1x_2=x_2x_1$, then $xfx\equiv xxf\in I_1$. Similarly if $\beta(f)=1$, by $x_1x_2=-x_2x_1$ we have $xfx\equiv -xxf\in I_1$.

(2) Let $\alpha(x)=0$. As above we may consider $\alpha(f_1)=\alpha(f_2)=1$. Thus $\alpha(f_1xf_2)=0$ and $xf_1xf_2\in I_1$. Analogously we treat $\alpha(x)=1$. □

Proposition 27. *If the monomial $f(x_1,\dots,x_m)\in T_G(M_{1,1}\bar{\otimes} E)$ then $f\in I_1$.*

Proof. We induct on $q=\deg f$ to show that if $f\notin I_1$ then $f(a_1,\dots,a_m)\neq 0$ for some $a_i\in M_{1,1}\bar{\otimes} E$. The case $q=1$ being obvious we suppose $q>1$ and write $f=hx_i$ for some i. If $h\in I_1$ we are done. So suppose $h\notin I_1$ and $i=1$, that is $f=hx_1$. Then $h(b_1,\dots,b_m)\neq 0$ for some $b_i\in M_{1,1}\bar{\otimes} E$.

Let $\deg_{x_1} f=d$. Choose the b_i such that the generators e_1, e_2, e_3, e_4 of E appear in neither of them. Write $f=f_1x_1f_2x_1\dots x_1f_dx_1$ where f_i are monomials not containing x_1. Thus $f_i=f_i(x_2,\dots,x_m)$. Let $f(x)$, $h(b)$ and so on, denote $f(x_1,\dots,x_m)$, $h(b_1,\dots,b_m)$, respectively. We consider case by case the homogeneous degree $(\alpha(x_1),\beta(x_1))$ of x_1.

Case 1. $(\alpha(x_1),\beta(x_1))=(0,0)$. Since $f_1(b)b_1f_2(b)b_1\dots b_1f_d(b)b_1\neq 0$ we have $f_1(b)f_2(b)\dots f_d(b)\neq 0$. Taking $a=\left(\begin{smallmatrix}1&0\\ 0&1\end{smallmatrix}\right)\bar{\otimes} 1$ we obtain

$$f(a,b_2,\dots,b_m)=f_1(b)af_2(b)a\dots af_d(b)a=f_1(b)\dots f_d(b)\neq 0.$$

Case 2. $(\alpha(x_1),\beta(x_1))=(1,1)$. Choose $a=\begin{pmatrix}0&e_1\\e_2&0\end{pmatrix}\bar{\otimes}1$, by Lemma 26 we obtain $d\leqslant 2$. If $d=1$ then $f(a,b_2,\dots,b_m)=h(b)a\neq 0$. If $d=2$ by Lemma 26 we have $\alpha(f_2)=0$. But $h(b)=f_1(b)b_1f_2(b)\neq 0$ thus $f_1(b)af_2(b)\neq 0$. If $\alpha,\ \beta\in E_0$ then $\begin{pmatrix}0&e_1\\e_2&0\end{pmatrix}\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}\begin{pmatrix}0&e_1\\e_2&0\end{pmatrix}=\begin{pmatrix}\beta e_1e_2&0\\0&\alpha e_2e_1\end{pmatrix}$. Thus $f(a,b_1,\dots,b_m)=f_1(b)af_2(b)a\neq 0$.

Case 3. $(\alpha(x_1),\beta(x_1))=(0,1)$. Let $a=\begin{pmatrix}1&0\\0&1\end{pmatrix}\bar{\otimes}e_1$, $a'=\begin{pmatrix}1&0\\0&e_2e_3\end{pmatrix}\bar{\otimes}e_4$. By Lemma 26 we have $d\leqslant 2$. If $d=1$ then $f(a,b_2,\dots,b_m)=h(b)a\neq 0$. If $d=2$ then $\alpha(f_2)=1$. But $h(b)=f_1(b)b_1f_2(b)\neq 0$, therefore $f_1(b)af_2(b)\neq 0$. Denote $f_2(b)=\sum_j\begin{pmatrix}0&\alpha_j\\\beta_j&0\end{pmatrix}\bar{\otimes}\gamma_j$, then $af_2(b)a'=-\sum_j\begin{pmatrix}0&\alpha_je_2e_3\\\beta_j&0\end{pmatrix}\bar{\otimes}e_1\gamma_je_4$ and $a'f_2(b)a=-\sum_j\begin{pmatrix}0&\alpha_j\\e_2e_3\beta_j&0\end{pmatrix}\bar{\otimes}e_4\gamma_je_1$. Thus we arrive at

$$f(a+a',b_2,\dots,b_m)=f_1(b)(a+a')f_2(b)(a+a')=f_1(b)a+f_2(b)a'+f_1(b)a'f_2(b)a$$

which equals $f_1(b)af_2(b)\begin{pmatrix}1-e_2e_3&0\\0&e_2e_3-1\end{pmatrix}\bar{\otimes}e_4\neq 0$.

Case 4. $(\alpha(x_1),\beta(x_1))=(1,0)$. Let $a=\begin{pmatrix}0&e_1\\e_2&0\end{pmatrix}\bar{\otimes}e_3$, then

$$f(a+b_1,b_2,\dots,b_m)=\sum_{i=1}^{d}f_1(b)b_1\dots f_i(b)af_{i+1}(b)b_1\dots f_d(b)b_1.$$

We first consider $\alpha(f_i)=1$ for every $i\geqslant 2$. If $d\geqslant p$ we obtain $f\in I_1$ by the identity $c(x_1,\dots,x_p)$. Thus $d<p$, and by using $x_1x_2x_3=x_3x_2x_1$ we arrive at $f(a+b_1,b_2,\dots,b_m)=df_1(b)bf_2(b)\dots b_1f_d(b)a=dh(b)a\neq 0$.

So let $\alpha(f_i)=0$ for some $i\geqslant 2$. Let t be the number of the j, $2\leqslant j\leqslant d$ such that $\alpha(f_jx_1f_{j+1}x_1\dots x_1f_d)=0$. Set r to be the largest j with this property. Denote $u=f_1(b)b_1f_2(b)b_1\dots b_1f_d(b)a$, $v=f_1(b)b_1f_2(b)b_1\dots b_1f_{r-1}(b)af_r(b)b_1\dots b_1f_d(b)b_1$. Then $x_1x_2x_3=x_3x_2x_1$ gives us $f(a+b_1,b_2,\dots,b_m)=(d-t)u+tv$. If $d-t\geqslant p$ and/or $t\geqslant p$, according to the identity c we have $f\in I_1$. An easy computation shows that $\begin{pmatrix}0&\alpha\\\beta&0\end{pmatrix}\begin{pmatrix}\gamma&0\\0&\delta\end{pmatrix}\begin{pmatrix}0&e_1\\e_2&0\end{pmatrix}=\begin{pmatrix}\alpha\delta e_2&0\\0&\beta\gamma e_1\end{pmatrix}$ and $\begin{pmatrix}0&e_1\\e_2&0\end{pmatrix}\begin{pmatrix}0&\alpha\\\beta&0\end{pmatrix}=\begin{pmatrix}e_1\delta\beta&0\\0&e_2\gamma\alpha\end{pmatrix}$. Thus e_1 and e_2 lie in different rows of the respective products for u and v. Hence u and v are linearly independent and $f(a+b_1,b_2,\dots,b_m)\neq 0$. $\square$

Proposition 27 and Theorem 23 yield immediately

Theorem 28. *Let* char $K=p\neq 2$. *The G-graded identities of $M_{1,1}\bar{\otimes}E$ follow from the set I_1.*

Now let E' be the nonunitary infinite dimensional Grassmann algebra, and let $A=M_{1,1}(E')\oplus K$ be the algebra obtained by $M_{1,1}(E')$ by means of formal adjoining an unit. Then $A=A_0\oplus A_1$ is 2-graded in the same way as $M_{1,1}$. Therefore $A\otimes E=A_0\otimes E_0\oplus A_0\otimes E_1\oplus A_1\otimes E_0\oplus A_1\otimes E_1$ is G-graded.

It is immediate that $I_1\subseteq T_G(A\otimes E)$, hence $T_G(M_{1,1}\bar{\otimes}E)\subseteq T_G(A\otimes E)$. Below we shall prove that this inclusion is proper whenever char $K=p>2$.

Let $f(y,z)=[y^p,z]$, $\alpha(y)=\beta(y)=0$ and $\alpha(z)=1$. We choose $y=\begin{pmatrix}1&0\\0&2\end{pmatrix}\bar{\otimes}1$, $z=\begin{pmatrix}0&g\\g&0\end{pmatrix}\bar{\otimes}h$ where $g\in E_1$, $h\in E_0\cup E_1$, $g,\ h\neq 0$. Then $[y^p,z]=\begin{pmatrix}0&(1-2^p)g\\(2^p-1)g&0\end{pmatrix}\bar{\otimes}h\neq 0$ since $2^p\equiv 2\neq 1 \pmod p$.

Now let $a=\sum_{i=1}^r\begin{pmatrix}e_i&0\\0&f_i\end{pmatrix}+k_i$, $e_i,\ f_i\in E_0'$, $k_i\in K$. Then $a^p=\sum_{i=1}^r\begin{pmatrix}e_i^p&0\\0&f_i^p\end{pmatrix}+k_i^p$. Since $x^p=0$ is an identity for E' we get $a^p=\sum_{i=1}^r k_i^p$. Take $b\in E_0$, then $a^p\bar{\otimes}b^p$ is central and hence $[y^p,z]=0$

in $A \otimes E$. Therefore $[y^p, z] \in T_G(A \otimes E)$. Thus we arrive at $T_G(M_{1,1} \bar{\otimes} E) \subsetneq T_G(A \otimes E)$. Therefore $T(M_{1,1} \bar{\otimes} E) \subseteq T(A \otimes E)$.

Now let $f(y,z) = [y^{p^2}, z]$. As above $y = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) \bar{\otimes} 1$, $z = \left(\begin{smallmatrix} 0 & g \\ g & 0 \end{smallmatrix}\right) \bar{\otimes} h$ where $g \in E_1$, $h \in E$, g, $h \neq 0$ yield that $f(y,z) \neq 0$. That is $[y^{p^2}, z] \notin T(M_{1,1} \bar{\otimes} E)$. It was proved in [3], that for every $a \in A$, the element a^{p^2} is central. Therefore $[(a \otimes e)^{p^2}, b \otimes f] = [a^{p^2} \otimes e^{p^2}, b \otimes f] = a^{p^2} b \otimes (e^{p^2} f - f e^{p^2})$. But $[y^{p^2}, z]$ is an identity for E thus $[y^{p^2}, z] \in T(A \otimes E)$. Therefore $T(M_{1,1} \bar{\otimes} E) \subsetneq T(A \otimes E)$. On the other hand one of the main results in [2] was that $T(A) = T(E \otimes E)$ therefore $T(M_{1,1} \bar{\otimes} E) \subsetneq T(E \otimes E \otimes E) = T(A \otimes E)$.

Moreover in [3] it was shown that the subalgebra B of the matrices in $M_2(E)$ whose entries on the second diagonal lie in E' satisfies $T_G(B) = I_1$. Therefore $T(M_2(E)) \subsetneq T(M_{1,1} \bar{\otimes} E)$. The example that shows the proper inclusion is exactly the same as that of [3]. Thus we have proved the following theorem.

Theorem 29. *Let* $\operatorname{char} K = p > 2$, *then* $T(M_2(E)) \subset T(M_{1,1} \bar{\otimes} E)$, *a proper inclusion.*

4. β-tensor products

Let A be a PI algebra and let $T(A)$ be its T-ideal. Denote by P_n the vector space of all multilinear polynomials in $x_1, \dots, x_n$ in the free associative algebra $K(X)$. The symmetric group S_n acts on the left on P_n by permuting the variables and $P_n \cong KS_n$ as S_n-modules. Also $T(A) \cap P_n$ is a submodule of P_n and one uses the representations of S_n in order to study PI algebras. The module $T(A) \cap P_n$ tends to be huge; the quotient $P_n(A) = P_n/(T(A) \cap P_n)$ is smaller and still keeps useful information about the identities of A. Of particular interest is the sequence of codimensions of A, $c_n(A) = \dim P_n(A)$. In [13] Regev proved that if A satisfies a polynomial identity then $c_n(A)$ is exponentially bounded, and this fact was the keystone in the proof of his $A \otimes B$ theorem, see [13]. Here we obtain a β-analog of Regev's $A \otimes B$ theorem.

Theorem 30. *Let K be a field and let G be a finite abelian group with a skew-symmetric bicharacter β. Let further A and B be two PI algebras that are also G-graded. Then* $c_n(A \otimes_\beta B) \leqslant |G|^{2n} c_n(A) c_n(B)$.

Proof. If an algebra satisfies a polynomial identity of degree d then it satisfies a multilinear identity of degree $\leqslant d$. Taking a consequence of it we may suppose that it satisfies a multilinear identity of degree d.

Let $g(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \gamma_\sigma x_{\sigma(1)} \dots x_{\sigma(n)}$ be a multilinear polynomial of degree n. We look for conditions on the coefficients $\{\gamma_\sigma \mid \sigma \in S_n\}$ which will assure that $g(x_1, \dots, x_n)$ is an identity for $A \otimes_\beta B$. Since g is multilinear it suffices to consider only substitutions of the type $x_k \mapsto a_k \otimes_\beta b_k$ where the elements a_k and b_k are homogeneous in the corresponding G-gradings. Then

$$
\begin{aligned}
g(a_1 \otimes_\beta b_1, \dots, a_n \otimes_\beta b_n) &= \sum_{\sigma \in S_n} \gamma_\sigma (a_{\sigma(1)} \otimes_\beta b_{\sigma(1)}) \dots (a_{\sigma(n)} \otimes_\beta b_{\sigma(n)}) \\
&= \sum_{\sigma \in S_n} \gamma_\sigma \left(\prod_{j=1}^{n-1} \prod_{i=j+1}^{n} \beta(h_{\sigma(j)}, g_{\sigma(i)}) \right) a_{\sigma(1)} \dots a_{\sigma(n)} \otimes_\beta b_{\sigma(1)} \dots b_{\sigma(n)}
\end{aligned}
$$

since β is a skew-symmetric bicharacter.

Let $c_n(A) = \alpha_n$ and $c_n(B) = \delta_n$ then $\dim P_n(A) = \alpha_n$, $\dim P_n(B) = \delta_n$. There exist α_n monomials in P_n, say $M = \{M_i(x_1, \dots, x_n) \mid 1 \leqslant i \leqslant \alpha_n\}$ such that for every $\sigma \in S_n$, the monomial $x_{\sigma(1)} \dots x_{\sigma(n)}$ is a combination of monomials in M modulo $T(A)$. Hence for every $\sigma \in S_n$ there exist $r_{\sigma,i} \in K$ with $a_{\sigma(1)} \dots a_{\sigma(n)} = \sum_{i=1}^{\alpha_n} r_{\sigma,i} M_i(a_1, \dots, a_n)$. Analogously we treat B. There exist δ_n monomials, say $\{N_j(x_1, \dots, x_n) \mid 1 \leqslant j \leqslant \delta_n\}$, and coefficients $s_{\sigma,j}$ such that for every $\sigma \in S_n$ we have $b_{\sigma(1)} \dots b_{\sigma(n)} = \sum_{j=1}^{\delta_n} s_{\sigma,j} N_j(b_1, \dots, b_n)$. Therefore denoting $M_i \otimes_\beta N_j = M_i(a_1, \dots, a_n) \otimes_\beta N_j(b_1, \dots, b_n)$ we have that $g(a_1 \otimes_\beta b_1, \dots, a_n \otimes_\beta b_n)$ equals

$$\sum_{i=1}^{\alpha_n}\sum_{j=1}^{\delta_n}\left[\sum_{\sigma\in S_n}\left(\prod_{j=1}^{n-1}\prod_{i=j+1}^{n}\beta(h_{\sigma(j)},g_{\sigma(i)})\right)r_{\sigma,i}s_{\sigma,j}\gamma_\sigma\right]M_i\otimes_\beta N_j.$$

Consider the $n!$ coefficients γ_σ as variables in the following linear system:

$$\left(\sum_{\sigma\in S_n}\left[\prod_{j=1}^{n-1}\prod_{i=j+1}^{n}\beta(h_{\sigma(j)},g_{\sigma(i)})\right]r_{\sigma,i}s_{\sigma,j}\right)\gamma_\sigma=0,\quad 1\leqslant i\leqslant\alpha_n,\ 1\leqslant j\leqslant\delta_n.$$

Since the value of $\beta(h_j,g_i)$ depends only on the G-degrees of a_i and b_j we will have at most $|G|^2$ possibilities for $\beta(h_j,g_i)$. Moreover $\deg g(x_1,\dots,x_n)=n$ hence we will have at most $(|G|^2)^n$ products $\prod_{j=1}^{n-1}\prod_{i=j+1}^{n}\beta(h_{\sigma(j)},g_{\sigma(i)})$, $\sigma\in S_n$. Thus there will be at most $|G|^{2n}\alpha_n\delta_n$ equations in our system. Since $c_n(A\otimes_\beta B)$ equals the rank of the system we obtain $c_n(A\otimes_\beta B)\leqslant|G|^{2n}c_n(A)c_n(B)$. $\square$

Theorem 31. *Let G be a finite abelian group and K a field. Suppose that A and B are PI algebras that are G-graded. Assume that β is a skew-symmetric bicharacter on G. Then $A\otimes_\beta B$ is again a PI algebra.*

Proof. According to [13], there exist constants P and Q such that $\alpha_n<P^n$ and $\delta_n<Q^n$ for every n. (If A satisfies an identity of degree d then one may take $P=(d-1)^2$, see for example [11, p. 111].) Therefore $c_n(A\otimes_\beta B)<R^n$ for some constant R, and for all sufficiently large n one has $c_n(A\otimes_\beta B)<n!$. Hence the above system admits nontrivial solutions for all sufficiently large n. But each such solution yields a polynomial identity for $A\otimes_\beta B$. $\square$

5. $M_n(K)\otimes_\beta E$ and $M_n(E)$

Assume that A and B are K-algebras, A is G-graded and B is H-graded for some additive abelian groups G and H. Let $\beta_A: G\times G\to K^*$ and $\beta_H: H\times H\to K^*$ be skew-symmetric bicharacters. Suppose that $a_{g_1}a_{g_2}=\beta_A(g_1,g_2)a_{g_2}a_{g_1}$ and $b_{h_1}b_{h_2}=\beta_B(h_1,h_2)b_{h_2}b_{h_1}$ for all $a_i\in A_{g_i}$, $b_j\in B_{h_j}$.

One defines a skew-symmetric bicharacter $\beta:(G\times H)\times(G\times H)\to K^*$ putting $\beta((g_1,h_1),(g_2,h_2))=\beta_A(g_1,g_2)\beta_B(h_1,h_2)$ and

$$(a_{g_1}\otimes_\beta b_{h_1})(a_{g_2}\otimes_\beta b_{h_2})=\beta\big((g_2,h_2),(g_1,h_1)\big)a_{g_1}a_{g_2}\otimes_\beta b_{h_1}b_{h_2}.$$

One sees easily that $(a_{g_1}\otimes_\beta b_{h_1})(a_{g_2}\otimes_\beta b_{h_2})=a_{g_2}a_{g_1}\otimes_\beta b_{h_2}b_{h_1}$.

Since

$$(a_{g_2}\otimes_\beta b_{h_2})(a_{g_1}\otimes_\beta b_{h_1})=\beta_A(g_1,g_2)\beta_B(h_1,h_2)a_{g_2}a_{g_1}\otimes_\beta b_{h_2}b_{h_1}$$

we obtain

$$\beta_A(g_2,g_1)\beta_B(h_2,h_1)(a_{g_2}\otimes_\beta b_{h_2})(a_{g_1}\otimes_\beta b_{h_1})=a_{g_2}a_{g_1}\otimes_\beta b_{h_2}b_{h_1}.$$

Therefore

$$(a_{g_1}\otimes_\beta b_{h_1})(a_{g_2}\otimes_\beta b_{h_2})=\beta\big((g_2,h_2),(g_1,h_1)\big)(a_{g_2}\otimes_\beta b_{h_2})(a_{g_1}\otimes_\beta b_{h_1})$$

thus the polynomial $[x_{(g_1,h_1)},x_{(g_2,h_2)}]_\beta$ is a graded identity for $A\otimes_\beta B$.

Let $G=\mathbb{Z}_n\times\mathbb{Z}_n$ and let ε be a primitive nth root of the unity. Set

$$A=\begin{pmatrix}\varepsilon^{n-1} & 0 & \dots & 0\\ 0 & \varepsilon^{n-2} & \dots & 0\\ & & \dots & \\ 0 & 0 & \dots & 1\end{pmatrix},\qquad B=\begin{pmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1\\ 1 & 0 & 0 & \dots & 0\end{pmatrix}\in M_n(K).$$

Then $AB=\varepsilon BA$, $A^n=B^n=1$. The matrices A^iB^j, $1\leqslant i,j\leqslant n$, are linearly independent and form a basis of the vector space $M_n(K)$.

If $g=(\bar r,\bar s)\in G$, $\bar r$, $\bar s\in\mathbb{Z}_n$ the homogeneous component of $M_n(K)$ of G-degree g is the span of A^rB^s. Then $M_n(K)=\bigoplus_{g\in G}(M_n(K))_g$, and $(M_n(K))_g(M_n(K))_h\subseteq(M_n(K))_{g+h}$, g, $h\in G$. Thus the above is a fine G-grading on $M_n(K)$. Define $\beta_1:G\times G\to K^*$ by $\beta_1(g,h)=\varepsilon^{rv-st}$ where $g=(\bar r,\bar s)$, $h=(\bar t,\bar v)\in\mathbb{Z}_n$. Then $C_gD_h=\beta_1(g,h)D_hC_g$ for every g, $h\in\mathbb{Z}_n$ and C_g, $D_h\in M_n(K)$. The identities for this grading were computed in [4].

Theorem 32. *(See [4].) Let $M_n(K)$ be G-graded as above. Then the polynomials*

$$[x_{g_1},x_{g_2}]_{\beta_1}=x_{g_1}x_{g_2}-\beta_1(g_1,g_2)x_{g_2}x_{g_1},\quad g_1,g_2\in G$$

form a basis of the G-graded identities for $M_n(K)$.

Denote by β_2 the skew symmetric bicharacter on $\mathbb{Z}_2$ defined as $\beta_2(\bar i,\bar j)=(-1)^{ij}$, then $\beta=\beta_1\beta_2$ is a skew symmetric bicharacter on $G\times\mathbb{Z}_2$.

Lemma 33. *In the notation introduced above the algebra $M_n(K)\otimes_\beta E$ is β-commutative. Furthermore no nonzero monomial of the type $x_{(g_1,i_1)}\dots x_{(g_n,i_n)}$, $g_j\in G$, $i_j\in\mathbb{Z}_2$, is a graded identity for $M_n(K)\otimes_\beta E$.*

Proof. For the first assertion it suffices to prove that $[x_{(g_1,i)},x_{(g_2,j)}]_\beta=0$ is an identity for $M_n(K)\otimes_\beta E$. This consists of a direct verification and we omit it. The second is also obvious. □

Corollary 34. *The ideal of the $G\times\mathbb{Z}_2$-graded identities for $M_n(K)\otimes_\beta E$ is generated by all commutators $[x_{(g_1,i)},x_{(g_2,j)}]_\beta$.*

Proof. The proof is similar to that given in [10], see also Theorem 10 for the $\mathbb{Z}_2$-graded case. □

On the other hand $M_n(E)$ is $G\times\mathbb{Z}_2$-graded in a natural way. If $(g,i)\in G\times\mathbb{Z}_2$ then its homogeneous component of degree (g,i) is $E_iA^rB^s$ where $g=(\bar r,\bar s)$. The same β defined above is a commutation factor for $M_n(E)$, that is $M_n(E)$ is β-commutative. One proves as in Lemma 33 that no nonzero monomial is a graded identity for $M_n(E)$. Furthermore as in Corollary 34 one has

Corollary 35. *The ideal of the $G\times\mathbb{Z}_2$-graded identities for $M_n(E)$ is generated by all commutators $[x_{(g_1,i)},x_{(g_2,j)}]_\beta$.*

Theorem 36. *The algebras $M_n(E)$ and $M_n(K)\otimes_\beta E$ satisfy the same $G\times\mathbb{Z}_2$-graded identities. Moreover they are PI equivalent in the ordinary sense.*

Combining Corollaries 34 and 35 one has the result about the graded identities. The PI equivalence of these two algebras hence follows.

Note added in proof

We learned that O.M. Di Vincenzo and V. Nardozza had proved a result similar to our Theorems 13 and 18. Their paper, “On a Regev–Seeman conjecture about $\mathbb{Z}_2$-graded tensor products” has recently been accepted in *Israel Journal of Mathematics*.

Acknowledgment

The authors thank Yu. Bahturin for his suggestions concerning the β-graded tensor product.

References

[1] S.M. Alves, P. Koshlukov, Polynomial identities of algebras in positive characteristic, J. Algebra 305 (2006) 1149–1165.
[2] S.S. Azevedo, M. Fidelis, P. Koshlukov, Tensor product theorems in positive characteristic, J. Algebra 276 (2) (2004) 836–845.
[3] S.S. Azevedo, M. Fidelis, P. Koshlukov, Graded identities and PI equivalence of algebras in positive characteristic, Comm. Algebra 33 (2005) 1011–1022.
[4] Yu. Bahturin, V. Drensky, Graded polynomial identities of matrices, Linear Algebra Appl. 357 (2002) 15–34.
[5] Yu. Bahturin, M. Zaicev, Graded algebras and graded identities, in: Polynomial Identities and Combinatorial Methods, in: Lect. Notes Pure Appl. Math., vol. 235, M. Dekker, 2003, pp. 101–139.
[6] A. Berele, Generic verbally prime algebras and their GK-dimensions, Comm. Algebra 21 (5) (1993) 1487–1504.
[7] A. Berele, Invariant theory and trace identities associated with Lie colour algebras, J. Algebra 310 (2007) 194–206.
[8] N. Bourbaki, Algèbre, III, Hermann, Paris, 1971.
[9] O.M. Di Vincenzo, V. Nardozza, Graded polynomial identities of verbally prime algebras, J. Algebra Appl. 6 (3) (2007) 385–401.
[10] O.M. Di Vincenzo, V. Nardozza, $\mathbb{Z}_{k+l} \times \mathbb{Z}_2$-graded polynomial identities for $M_{k,l}(E) \otimes E$, Rend. Sem. Mat. Univ. Padova 108 (2002) 27–39.
[11] V. Drensky, Free Algebras and PI-Algebras, Graduate Course in Algebra, Springer, Singapore, 1999.
[12] A. Kemer, Ideal of identities of associative algebras, Amer. Math. Soc. Transl. Ser. 87 (1991).
[13] A. Regev, Existence of identities in $A \otimes B$, Israel J. Math. 11 (1972) 131–152.
[14] A. Regev, T. Seeman, $\mathbb{Z}_2$-graded tensor products of PI-algebras, J. Algebra 291 (2005) 274–296.
[15] M. Scheunert, The Theory of Lie Superalgebras, Lecture Notes in Math., vol. 716, Springer, Berlin, 1979.
[16] S.Y. Vasilovsky, $\mathbb{Z}_n$-graded polynomial identities of the full matrix algebra of order n, Proc. Amer. Math. Soc. 127 (12) (1999) 3517–3524.
[17] C.T.C. Wall, Graded Brauer groups, J. Reine Angew. Math. 213 (1963/1964) 187–199.
[18] A.A. Zolotykh, Commutations factors and varieties of associative algebras, Fundam. Prikl. Mat. 3 (2) (1997) 453–468 (in Russian).